Time Value of Money

Mathematics of Finance
Compounding and Discounting

Reasons for interest

Lender’s side

- Reward for postponing consumption
- Compensation for risk
  - Default risk
  - Purchasing power risk (inflation)
  - Liquidity risk

Borrower’s side

- Productivity of capital
- Reinvest the funds at a higher rate

Mathematics of finance

P_0 = principal at time 0
S_t = future sum at time t
n = number of compounding **years**
i = interest rate **per year**

Lump-sum compounding

$S_1 = P_o + P_0 i$
$S_2 = S_1 + S_1 i$
$S_2 = P_0 (1 + i)^2$
$S_n = P_0 (1 + i)^n$
$(1 + i)^n = (FVIF – i\% - n)$
(FVIF – i% - n) = Future Value Interest Factor for i% and n years

Simple example

If P_0 = \$25, n = 5 and i = 6%
$S_5 = 25(1.06)^5 = 33.46$
S_5 = 25(FVIF – 6% - 5)
$S_5 = 25(1.3382) = 33.46$

Using a financial calculator:
25➔PV 6➔I/yr 5➔n FV=33.46

\$25 invested today at 6% will grow to \$33.46 in 5 years

Frequency of compounding

Bonds	Semiannually	2 times/yr
Savings Accts	Quarterly	4 times/yr
Car Loans & Mortgages	Monthly	12 times/yr
MC/Visa	Daily	365 times/yr

Quarterly compounding

$S_n = P_0 (1 + i)^n$

i = interest rate **per period**

n = number of **periods**

Passbook offers 8%/yr comp quarterly

i = 2%/period and n = 4 periods/yr

$S_{1Q} = P_0(1.02)$

$S_{2Q} = P_0(1.02)(1.02)$

$S_{4Q/1yr} = P_0(1.02)^4$

Effective Annual Rate

$$EAR = \frac{\$Interest}{Principal} = \frac{S_{1yr} - P_0}{P_0}$$

$$EAR = \frac{P_0(1.02)^4 - P_0}{P_0} = (1.02)^4 - 1 = 8.24\%/yr$$

$EAR = (1+i)^n - 1$ <<== KEY!!

where :

i = interest rate per period

n = number of periods in a year

Why EAR?

Two alternative investments:

A: APR = 21%/ yr compounded semiannually

B: APR = 20%/ yr compounded daily

$$EAR_A = (1+\frac{.21}{2})^2 - 1 = 22.10\%/yr$$

$$EAR_B = (1+\frac{.20}{365})^{365} - 1 = 22.13\%/yr$$

Car loan example

Dealer offers financing at 12%/year, compounded monthly

What rate are they really charging?

$EAR = (1 + .01)^{12} - 1 = 12.68\%$

Discounting and present value

Reciprocals of compounding and future value

$33.46 to be paid in 5 yrs is worth how much today if the interest rate is 6%/yr?

$S_n = P_0(1 + i)^n$

$P_0 = S_n/(1 + i)^n = 33.46/(1.06)^5$

$1/(1+i)^n = (PVIF - i\% - n)$

(PVIF –i% - n) = Present Value Interest Factor for i% and n periods

Solution (Cont'd)

$P_0 = 33.46/(1.06)^5 = 33.46(PVIF\text{-}6\%\text{-}5) = 25.00$

(.7473)

Using a financial calculator:

33.46➔FV 6➔I/yr 5➔n PV = 25.00

$25 invested today at 6% will grow to $33.46 in 5 years

Same example, different frequency

Assume 6%/yr compounded semiannually so now i = 3% a period

Still 5 years so now n = 10 periods

$P_0 = 33.46/(1.03)^{10} = 33.46(\text{PVIF-3\%-10}) = 24.90$

Find EAR: EAR = $(1.03)^{??} - 1$

What's the ??

It's NOT 10

It's EAR = $(1.03)^2 - 1$ periods = 2, not 10

Remember it's E**A**R and the **A** is "annual" and there are 2 periods in a year if it's semiannual compounding

Irrelevant that it's a 5 year investment

Why we need the time value of money

Two gifts from your rich uncle:

A:

	100	100	100			250
0	1	2	3	4	5	6

B:

					325	325
0	1	2	3	4	5	6

What else do we need to know in order to decide?

Important missing piece

- Who is the guy? Mom says it's her brother but can you be sure?
- Expected inflation rate over the next 6 years?
- How much do we need money in the next couple of years?
- How much can we sell the gifts for now?
- Assume an interest rate of i = 10%

Which gift is worth more?

$$PV_{A,0} = \frac{100}{(1.10)^1} + \frac{100}{(1.10)^2} + \frac{100}{(1.10)^3} + \frac{250}{(1.10)^6} = 389.80$$

$$PV_{B,0} = \frac{325}{(1.10)^5} + \frac{325}{(1.10)^6} = 385.25$$

$$FV_{A,6} = 100(1.10)^5 + 100(1.10)^4 + 100(1.10)^3 + 250 = 690.56$$

$$FV_{B,6} = 325(1.10)^1 + 325 = 682.50$$

$$\text{Note that } \frac{690.56}{(1.10)^6} = 389.80$$

What happens to the $389.80?

Time	Inflow	Interest	Outflow	Value
0	389.80	-	-	389.80
1	-	38.98	-100	328.78
2	-	32.88	-100	261.66
3	-	26.17	-100	187.83
4	-	18.78	-	206.61
5	-	20.66	-	227.27
6	-	22.73	-250	0.00

Observations

You could duplicate your uncle's gift by investing $389.80 for 6 years at 10%

You could sell your uncle's gift to your brother today for $389.80 and he would earn 10%

If the interest rate were low, say 2%, then B is a lot more attractive than A

If the interest rate were high, say 50%, then A is a lot more attractive than B

Annuities

Constant amounts, regular fixed intervals

Series of equal amounts, received or paid, at regular constant intervals

Ordinary annuity ➔ payments are at the ***end*** of each period. **Annuity begins one period prior to the first payment**

Present Value of an Annuity

$$A \underset{0}{\rule{1em}{0.4pt}} \underset{1}{R} \rule{1em}{0.4pt} \underset{2}{R} \rule{1em}{0.4pt} \underset{3}{R} // \underset{n-1}{R} \rule{1em}{0.4pt} \underset{n}{R}\; B$$

$$PV_A = \frac{R}{(1+i)^1} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}$$

$$PV_A = R[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n}]$$

$$PV_A = R[\frac{(1+i)^n - 1}{i(1+i)^n}]$$

$$[\frac{(1+i)^n - 1}{i(1+i)^n}] = [PVIF_a - i\% - n]$$

($PVIF_a$ - i% - n)

($PVIF_a$-i%-n) is the present value interest factor of an annuity of $1.00 per period for n periods discounted at i% per period

It is a commonly used short-hand notation

PV of annuity example

Find the PV of a 10 year annuity that pays $50 every six months. Use an interest rate of 6% a year, compounded semiannually

PV = 50(PVIFa – 3% - 20)

Using a financial calculator:

50➔PMT 3➔I/yr 20➔n PV=$743.87

Monthly car payments

Buy a car for $15,000 by putting $5,000 down and borrowing $10,000 from dealer. It is a 4 year loan with monthly payments.

Interest rate is 12%/yr, compounded monthly

10,000 = R($PVIF_a$-1%-48)

10000➔PV 1➔I/yr 48➔n PMT=$263.34

Deferred Annuity

			10	10	10
0	1	2	3	4	5

$i = 5\%$

First find $PV_2 = 10(PVIF_a\text{-}5\%\text{-}3) = 27.23$

Then discount the 27.23 back two more periods

$$PV_0 = \frac{10(PVIF_a - 5\% - 3)}{(1.05)^2} = 24.70$$

Perpetual Annuity

You have $200 at time 0.

You invest it for 1 period at 10%/period

You now have 220 = 200 (1.10)

You withdraw the 20 interest payment leaving you with the original 200 principal

You invest it for another period at 10%

You now have 220 = 200 (1.10)

You withdraw the 20 interest payment leaving you with the original 200 principal

You can continue to do this for ever if you do not touch the original principal

Perpetual Annuity

PV=200 and i=.10, then R=(200)(.10)=20

If R = (PV)(i), then PV = R/i

$20/period for n→8 discounted at 10% is PV=20/.10=200

Deferred Perpetual Annuity

0	1	2	3	4	5	6	//	∞
					50	50		50

Assume an interest rate of 5%

$$PV_4 = \frac{50}{.05} = 1000$$

$$PV_0 = \frac{1000}{(1.05)^4} = 822.70$$

$$PV_0 = \frac{\left[\frac{50}{.05}\right]}{(1.05)^4} = 822.70$$

Future Value of an Annuity

A	0	1	2	3	//	n - 1	n	B
		R	R	R		R	R	

$$FV_B = R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots + R(1+i)^1 + R$$

$$FV_B = R[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^1 + 1]$$

$$FV_B = R[\frac{(1+i)^n - 1}{i}]$$

$$\left(\frac{(1+i)^n - 1}{i}\right) = (FVIF_a - i\% - n)$$

$(FVIF_a$ - i% - n)

$(FVIF_a$-i%-n) is the future value interest factor of an annuity of $1.00 per period for n periods compounded at i% per period

It is a commonly used short-hand notation

Sinking fund example

Goal is to save $10,000,000 in 10 years by making 10 equal annual deposits into sinking fund that pays 12% interest. First deposit is in one year. Find annual deposit.

FV=10,000,000=R($FVIF_a$-12%-10)

Using a financial calculator:

10000000➔FV 12➔I/yr 10➔n PMT=569,841.64

Sinking fund (cont'd)

What if firm can deposit only $500,000 per year for 10 years? Must earn higher than 12% to achieve $10,000,000 goal. Find i.

500000($FVIF_a$-i%-10)=10000000

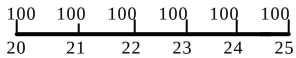
Using a financial calculator:

500000➔PMT 10➔n –10000000➔FV

i=14.69%

Putting it all together

- Your uncle gives you $100 today, your 20th birthday. He promises to give you $100 on your 21st, 22nd, 23rd, 24th and 25th birthdays as well. You invest all gifts in a savings acct paying 5% interest in order to someday buy a new stereo.
- On your 23rd birthday, your old stereo dies. Your brother offers you a lump sum on that day if you sign over to him the two remaining gifts (24th and 25th birthdays) when they come in but he wants a 12% return for his generosity.
- What's the most expensive stereo you can buy on your 23rd birthday using your savings and your brother's advance?



$$\text{Stereo} = 100(FVIF_a - 5\% - 4) + 100(PVIF_a - 12\% - 2)$$

$$\text{Stereo} = 431.01 + 169.01 = \$600.02$$